MATH2040 Linear Algebra II

Tutorial 7

October 27, 2016

1 Examples:

Example 1

Let $V = \mathbb{R}^4$ be an inner product space with the inner product $\langle x, y\rangle = \sum_{i=1}^{4} x_i y_i$, and let S be a subset of V defined by

$$S = \left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \end{pmatrix} \right\}.$$

(a) Apply the Gram-Schmidt process to S to obtain an orthogonal basis for $\text{span}(S)$.

(b) Normalize the above basis to obtain an orthonormal basis for $\text{span}(S)$.

(c) Express $z = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ as linear combination of the orthonormal basis in (b).

Solution

(a) Let $\{v_1, v_2, v_3\}$ be the orthogonal basis obtained from the Gram-Schmidt process, by formula we know the i-th element in the orthogonal basis is

$$v_i = s_i - \sum_{k=1}^{i-1} \frac{\langle s_i, v_k\rangle}{\langle v_k, v_k\rangle} v_k,$$

where s_i is the i-th vector in S.

Then we can find the orthogonal basis as follows:

(1)

$$v_1 = s_1 = (1, -2, -1, 3)^t$$

(2)

$$\begin{aligned} \langle v_1, v_1\rangle &= (1, -2, -1, 3)(1, -2, -1, 3)^t = 15 \\ \langle s_2, v_1\rangle &= (3, 6, 3, -1)(1, -2, -1, 3)^t = -15 \\ \text{So} \quad v_2 &= s_2 - \frac{-15}{15} v_1 = (4, 4, 2, 2)^t \end{aligned}$$

(3)

$$\begin{aligned}
\langle v_1, v_1\rangle &= (1,-2,-1,3)(1,-2,-1,3)^t = 15\\
\langle s_3, v_1\rangle &= (1,4,2,8)(1,-2,-1,3)^t = 15\\
\langle v_2, v_2\rangle &= (4,4,2,2)(4,4,2,2)^t = 40\\
\langle s_3, v_2\rangle &= (1,4,2,8)(4,4,2,2)^t = 40\\
\text{So} \quad v_3 &= s_3 - \frac{15}{15}v_1 - \frac{40}{40}v_2 = (-4,2,1,3)^t
\end{aligned}$$

(b) Note

$$\begin{aligned}
\langle v_1, v_1\rangle &= (1,-2,-1,3)(1,-2,-1,3)^t = 15\\
\langle v_2, v_2\rangle &= (4,4,2,2)(4,4,2,2)^t = 40\\
\langle v_3, v_3\rangle &= (-4,2,1,3)(-4,2,1,3)^t = 30
\end{aligned}$$

So the orthonormal basis $\{u_1, u_2, u_3\} = \{\frac{v_1}{\sqrt{15}}, \frac{v_2}{\sqrt{40}}, \frac{v_3}{\sqrt{30}}\}$

(c) We first find the following inner products:

$$\begin{aligned}
\langle z, v_1\rangle &= (-1,2,1,1)(1,-2,-1,3)^t = -3\\
\langle z, v_2\rangle &= (-1,2,1,1)(4,4,2,2)^t = 8\\
\langle z, v_3\rangle &= (-1,2,1,1)(-4,2,1,3)^t = 12
\end{aligned}$$

Then, by theorem we have

$$z = \sum_{i=1}^{3} \frac{\langle z, v_i\rangle}{\langle v_i, v_i\rangle} v_i = \frac{-3}{\sqrt{15}}u_1 + \frac{8}{\sqrt{40}}u_2 + \frac{12}{\sqrt{30}}u_3$$

Example 2

Let $S = \{\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\}$ in $\mathbb{C}^3$ with the inner product $\langle x, y\rangle = \sum_{i=1}^{3} x_i \overline{y_i}$. Compute $S^\perp$.

Solution

By definition

$$S^\perp = \{x \in \mathbb{C}^3 : \langle x, (1,0,i)^t\rangle = \langle x, (1,2,1)^t\rangle = 0\}.$$

So we first let $x = (a,b,c)^t$, where $a, b, c \in \mathbb{C}$. Note x must satisfy the orthogonal conditions:

$$\begin{cases} (a,b,c)(1,0,-i)^t = 0 \\ (a,b,c)(1,2,1)^t = 0 \end{cases}$$

i.e.

$$\begin{cases} a - ic = 0 \\ a + 2b + c = 0 \end{cases}$$

So $S^\perp = \text{span}\left\{\begin{pmatrix} i \\ -\frac{1+i}{2} \\ 1 \end{pmatrix}\right\}$.

Example 3

Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y\rangle \neq 0$.

Solution

Note x can be uniquely written as $x = u + v$, where $u \in W, v \in W^\perp$. Since $x \notin W$, so $v \neq 0$. Therefore, take $y = v \in W^\perp$, we have

$$\langle x, y\rangle = \langle u + v, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \langle v, v\rangle > 0.$$

Example 4

Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$.

Solution

On one hand, suppose $x \in (W_1 + W_2)^\perp$, then $\langle x, u + v\rangle = 0 \quad \forall u \in W_1, v \in W_2$. In particular, $\langle x, u\rangle = \langle x, v\rangle = 0 \quad \forall u \in W_1, v \in W_2$ since $0 \in W_1$ and $0 \in W_2$. So $x \in W_1^\perp$ and $x \in W_2^\perp$, therefore, $x \in W_1^\perp \cap W_2^\perp$.

On the other hand, suppose $x \in W_1^\perp \cap W_2^\perp$, then $\langle u, x\rangle = \langle v, x\rangle = 0 \quad \forall u \in W_1, v \in W_2$. By the linearity in the first argument, we have $\langle u + v, x\rangle = 0 \quad \forall u \in W_1, v \in W_2$. Therefore, $x \in (W_1 + W_2)^\perp$.

Example 5

Let V be a finite-dimensional inner product space over $\mathbb{F}$ and let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in V$, prove that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.$$

Solution Note x and y can be expressed as $x = \sum_{i=1}^{n} \langle x, v_i\rangle v_i$ and $y = \sum_{k=1}^{n} \langle y, v_k\rangle v_k$. Then,

$$\begin{aligned}
\langle x, y\rangle &= \langle \sum_{i=1}^{n} \langle x, v_i\rangle v_i, \sum_{k=1}^{n} \langle y, v_k\rangle v_k\rangle \\
&= \sum_{i=1}^{n} \langle x, v_i\rangle \langle v_i, \sum_{k=1}^{n} \langle y, v_k\rangle v_k\rangle \\
&= \sum_{i=1}^{n} \langle x, v_i\rangle \langle v_i, \langle y, v_i\rangle v_i\rangle \\
&= \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle} \langle v_i, v_i\rangle \\
&= \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}.
\end{aligned}$$

2 Exercises:

Question 1 (Section 6.2 Q2(g)):

Let $V = M_{2\times 2}(\mathbb{R})$ be an inner product space with the inner product $\langle A, B\rangle = tr(B^t A)$, and S be a subset of V defined by

$$S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}.$$

(a) Apply the Gram-Schmidt process to S to obtain an orthogonal basis for span(S).

(b) Normalize the above basis to obtain an orthonormal basis for span(S).

(c) Express $A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$ as linear combination of the orthonormal basis in (b).

Question 2 (Section 6.2 Q9):

Let $W = \text{span}\{(i, 0, 1)^t\}$ in $\mathbb{C}^3$. Find orthonormal bases for W and $W^\perp$.

Question 3 (Section 6.2 Q12):

Prove that for any matrix $A \in M_{m\times n}(\mathbb{F})$,

$$(R(L_{A^*}))^\perp = N(L_A).$$

Answers:

Question 1:

(a) Denote $\{s_1, s_2, s_3\}$ to be the corresponding matrices in S and $\{v_1, v_2, v_3\}$ to be the orthogonal basis. Then,

(1)

$$v_1 = s_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$$

(2)

$$\begin{aligned} &\langle v_1, v_1\rangle = 36 \\ &\langle s_2, v_1\rangle = 36 \\ &\text{So} \quad v_2 = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \end{aligned}$$

(3)

$$\begin{aligned} &\langle v_1, v_1\rangle = 36 \\ &\langle s_3, v_1\rangle = -72 \\ &\langle v_2, v_2\rangle = 72 \\ &\langle s_3, v_2\rangle = -72 \\ &\text{So} \quad v_3 = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix} \end{aligned}$$

(b) Denote $\{u_1, u_2, u_3\}$ to be the orthonormal basis, we have

$$u_1 = \frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, u_2 = \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, u_3 = \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$$

(c) $A = 4v_1 + v_2 - v_3 = 24u_1 + 6\sqrt{2}u_2 - 9\sqrt{2}u_3$.

Question 2:

The orthonormal basis for W is $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \right\}$.

The orthonormal basis for $W^\perp$ is $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Question 3:

Since $R(L_{A^*}) = \{y \in \mathbb{F}^n : y = A^*x \quad \forall x \in \mathbb{F}^m\}$ is the range of L_{A^*}, so the corresponding orthogonal complement is $(R(L_{A^*}))^\perp = \{z \in \mathbb{F}^n : \langle z, y\rangle = 0 \quad \forall y \in R(L_{A^*})\}$. Note,

$$\begin{aligned} (R(L_{A^*}))^\perp &= \{z \in \mathbb{F}^n : \langle z, y\rangle = 0 \quad \forall y \in R(L_{A^*})\} \\ &= \{z \in \mathbb{F}^n : \langle z, A^*x\rangle = 0 \quad \forall x \in \mathbb{F}^m\} \\ &= \{z \in \mathbb{F}^n : \langle Az, x\rangle = 0 \quad \forall x \in \mathbb{F}^m\} \\ &= \{z \in \mathbb{F}^n : Az = 0\} \\ &= N(L_A) \end{aligned}$$